THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 26, 1901, extending through the usual morning and afternoon sessions. Vice-President Professor Thomas S. Fiske occupied the chair. The American Physical Society met on the same day, and the mutual interests of the two societies were renewed by a joint session, presided over by President A. A. Michelson of the Physical Society, at which Professor J. Hadamard, the official delegate of the University of Paris at the Yale bicentennial celebration, read a paper entitled "On the theory of elastic plates."

The total attendance at both sessions exceeded forty persons, including the following thirty-three members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Dr. L. P. Eisenhart, Dr. William Findlay, Professor H. B. Fine, Professor T. S. Fiske, Miss Ida Griffiths, Miss Carrie Hammerslough, Dr. G. W. Hill, Dr. A. A. Himowich, Professor Harold Jacoby, Dr. Edward Kasner, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James Maclay, Mr. H. B. Mitchell, Dr. I. E. Rabinovitch, Professor J. K. Rees, Mr. C. H. Rockwell, Miss I. M. Schottenfels, Professor C. A. Scott, Professor D. E. Smith, Dr. H. F. Stecker, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor L. A. Wait, Professor A. G. Webster, Miss E. C. Williams, Dr. Ruth G. Wood, Professor R. S. Woodward.

The Council announced the election of the following twelve persons to membership in the Society: Mr. C. H. Ashton, Harvard University; Professor H. Y. Benedict, University of Texas; Dr. William Findlay, Columbia University; Dr. W. B. Fite, Cornell University; Professor G. W. Greenwood, McKendree College; Professor F. W. Hanawalt, Iowa Wesleyan University; Dr. E. V. Huntington, Harvard University; Professor H. W. Kuhn, Ohio State University; Dr. I. E. Rabinovitch, New York, N. Y.; Professor W. D. Tallman, Montana State College of Agriculture and Mechanic Arts; Mr. H. M. Tory, McGill Uni-

versity; Mr. A. H. Wilson, Princeton University. Seven applications for membership were received.

Upon recommendation of the Council, By-Law VII was amended to provide that the presidential address shall hereafter be delivered at the annual meeting at which the presidential term expires. As this amendment goes into effect immediately, the next presidential address will occur at the annual meeting in December, 1902.

The following papers were presented at the meeting:

(1) Professor G. A. MILLER: "On the abelian groups which are conformal with non-abelian groups."

(2) Dr. H. F. STECKER: "Concerning the elliptic $\wp(g_2; g_3; z)$-functions as coördinates in a line complex, and certain related theorems."

(3) Miss I. M. SCHOTTENFELS: "Generational definitions of certain groups of order 960."

(4) Professor OTTO STOLZ: "Zur Erklärung der Bogenlänge und des Inhaltes einer krümmen Fläche."

(5) Dr. L. P. EISENHART: "Conjugate rectilinear congruences."

(6) Dr. S. E. SLOCUM: "The symbols of the infinitesimal transformations which generate the parameter groups corresponding to all possible types of structure of two, three, and four parameter complex groups."

(7) Dr. E. V. HUNTINGTON and Mr. J. K. WHITTEMORE: "Some curious properties of conics touching the line infinity at one of the circular points."

(8) Professor J. HADAMARD: "On the theory of elastic plates."

(9) Professor E. B. VAN VLECK: "On the zeros of fundamental integrals of regular linear differential equations of the second order, with a determination of the number of imaginary roots of the hypergeometric series."

(10) Dr. E. J. WILCZYNSKI: "Reciprocal systems of linear differential equations."

(11) Dr. I. E. RABINOVITCH: "On some contradictions involved in the elliptic geometry in a point space."

(12) Dr. EDWARD KASNER: "Determination of the integrals in the calculus of variations leading to an assigned system of extremals."

Professor Hadamard was introduced by Vice-President Fiske. Professor Stolz's paper was presented to the Society through Professor Moore. In the absence of the authors, Professor Miller's paper was read by Dr. Ling, and the papers of Professor Stolz, Dr. Slocum, Dr. Huntington and

Mr. Whittemore, and Dr. Wilczynski were read by title. The papers of Dr. Miller, Dr. Stecker, and Dr. Huntington and Mr. Whittemore will appear in the BULLETIN. Abstracts of the papers are given below.

Two distinct groups are said to be conformal when they contain the same number of operators of each order. Professor Miller's paper is devoted to the determination of all the abelian groups which are conformal with non-abelian groups. The results are as follows: The necessary and sufficient conditions that any abelian group of order $2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots$ ($p_1, p_2, \cdots$ being distinct odd primes) is conformal with at least one non-abelian group are: 1° At least one of its subgroups of orders $2^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$ is noncyclic; 2° if the order $p_\beta^{\alpha_\beta}$ of this subgroup is odd then $\alpha_\beta > 2$, if the order is even (2^{α_0}) then the subgroup must include operators of order 4 and $\alpha_0 > 3$.

Miss Schottenfels's paper treats of the generational definitions of certain abstract groups of order 960 containing an invariant abelian subgroup of order 16 and an icosahedral subgroup of order 60, with factors of composition 60, 2, 2, 2, 2, together with the determination of the generators of the substitution and Galois field groups holoedrically isomorphic with the above abstract groups.

Professor Stolz's paper, which will appear in the *Transactions*, gives a rigorous demonstration of the theorem that the integral

$$\int_\alpha^\beta \sqrt{\varphi'(t)^2 + \psi'(t)^2}\, dt$$

satisfies the definition which he has established for the length of the curve

$$x = \varphi(t), \quad y = \psi(t) \qquad (\alpha \leqq t \leqq \beta)$$

also in the case that the interval (α, β) may be divided into a finite number of parts on every one of which the derivatives $\varphi'(t), \psi'(t)$ are continuous. He further defines the area or content of a surface and indicates the course of the proof of the theorem analogous to that just mentioned.

Certain general formulæ, established by Cifarelli, showing the relations existing between the Kummer functions of a rectilinear congruence referred to a general system of parameters, are applied by Dr. Eisenhart to several important

cases, afforded by taking for the double family of parametric ruled surfaces in turn the principal ruled surfaces, the mean ruled surfaces, and one family of developables and their orthogonal trajectories. This application is made for the determination of all congruences having a given spherical representation of each one of these three double families of ruled surfaces. In each case it is found that the abscissa, measured from the surface of reference, of the point where the line of shortest distance between the lines (u, v) and $(u + du, v)$ meets the former satisfies a partial differential equation of the second order, whose coefficients are functions of the coefficients, and their derivatives, of the fundamental quadratic differential form of the sphere. And when this abscissa has been found, the further determination of the congruence reduces to the solution of a Riccati equation and quadratures.

These general results are then applied to the particular case where the given spherical representation of the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ consists of a family of great circles and their orthogonal trajectories; this particular choice is made throughout the remainder of the paper. After it is shown that the line of shortest distance between the lines (u, v) and $(u + du, v)$ is different from the line between the latter and its consecutive one in the ruled surface $v = \text{const.}$ of which all three are generatrices, it follows that these lines of shortest distance form a congruence, which is called the *conjugate* of the original. It is evident from this definition that, with the choice of another system of great circles and their orthogonal trajectories for the representation of the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ of the original congruence, there is a different conjugate congruence. Hence to each congruence there corresponds an indefinite number of conjugate congruences. Further, it can be shown that reciprocally the original congruence is a conjugate of each of its conjugates.

The above definition fails where the ruled surfaces $v = \text{const.}$ are developable; but another definition consistent with the results in the general case permits the consideration of this exceptional case.

Certain simple relations are readily found to exist between the Kummer functions for a congruence and a conjugate, from which can be determined the equation of condition that a conjugate congruence of a normal congruence be itself normal. It is found that the only surfaces whose normals form a congruence which has for a conjugate the aggregate of the tangents to the v lines on the surface are those for

which the spherical representation of one system of asymptotic lines, $v = \text{const.}$, is a family of geodesic parallels. When the original congruence is defined as the intersection of two planes, the corresponding conjugate congruence can be found by algebraic operations.

The problem of finding the direction cosines of a congruence for which $u = \text{const.}$, $v = \text{const.}$ is a double family of great circles and their orthogonal trajectories is the same as the determination of a skew curve from its intrinsic equation, which in general reduces to the integration of a Riccati equation and quadratures.

A further investigation of congruences and particular ones of their conjugates shows that where a congruence and its conjugate corresponding to the developables of the former are normal, their developable surfaces correspond, and further that all such congruences can be determined by quadratures; the complete determination of normal congruences whose mean ruled surfaces $u = \text{const.}$, $v = \text{const.}$ correspond to the mean ruled surfaces of the corresponding conjugate congruence reduces to quadratures, and further this conjugate is not a normal congruence.

Professor Hadamard presented his views on the theory of thin elastic plates, the manner in which the equation of the problem is to be obtained, and the different methods which have been proposed for this purpose. Especially as to the boundary conditions, different methods have been employed which lead to apparently opposite results. The explanation of the discrepancy lies in the properties of the integrals of the equation $\Delta u = k^2 u$, when k is a large number. It is easily recognized that in this case the function u, defined within a given area by the preceding equation and by boundary values given once for all, is indefinitely small at any sensible distance from the boundary. More precisely, if δ is the distance of any interior point M from the boundary and A the maximum of the absolute values of the given values of u on the boundary, we have

$$|u_M| < \frac{A}{J(ik\delta)},$$

J being the well-known Bessel function, which increases very rapidly when the argument increases by purely imaginary values, so that the second member of the above inequality is very small, when k is very large, as soon as δ has a finite value.

The author is thus led to consider the formula for the asymptotic representation of the Bessel function $J(x)$ for large values of x. This formula furnishes a *formal* expression for J by means of a divergent series. It would be of interest to apply to this case Borel's methods relative to such series. The following method might also be tried: Given a sequence of numbers $a_0, a_1, \cdots, a_n, \cdots$, such that the series $a_0 + a_1x + \cdots + a_nx^n + \cdots$ is divergent for any value of x, consider the series

$$a_0[1 - e^{-1/x} P_0(1/x)] + a_1x[1 - e^{-1/x} P_1(1/x)] + \cdots + a_nx^n[1 - e^{-1/x} P_n(1/x)] + \cdots,$$

where $P_1, P_2, \cdots, P_n, \cdots$ are polynomials, viz., the beginnings of the development of $e^{1/x}$ in powers of $1/x$. As in Weierstrass's proof of Mittag-Leffler's theorem, it will be easy to prove that the degrees of these successive polynomials can be so chosen that the above series shall be convergent for every sufficiently small value of x and represent a function whose asymptotic development in powers of x is precisely $a_0 + a_1x + \cdots + a_nx^n + \cdots$.

Professor Van Vleck's paper will appear in the January number of the *Transactions*.

Dr. Wilczynski's paper is in abstract as follows: Let y_k and z_k for $k = 1, 2, 3, 4$ be four pairs of simultaneous solutions forming a fundamental system of the differential equations

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

with the independent variable x.

If y_k and z_k are interpreted as homogeneous coördinates of two points, the integration of (1) gives two curves in space. If corresponding points, *i. e.*, points corresponding to the same value of x, of the two curves are joined by straight lines, the ruled surface formed by these lines remains invariant for all transformations of the form

$$(2) \quad x = f(\xi), \quad y = \alpha(\xi)\eta + \beta(\xi)\zeta, \quad z = \gamma(\xi)\eta + \delta(\xi)\zeta,$$

where f, α, β, γ, δ are arbitrary functions.

At corresponding points of the two curves, construct planes tangent to the ruled surface. Let their coördinates

be u_s and v_s respectively. The author sets up a system of the same form as (1) satisfied by u_s and v_s and calls it the system adjoined to (1). This system is set up in several different forms, the most convenient being

$$
\begin{aligned}
&U'' + p_{11}U' + p_{12}V' + [q_{11} + \tfrac{1}{4}(u_{11} - u_{22})]\,U \\
&\qquad\qquad + (q_{12} + \tfrac{1}{2}u_{12})\,V = 0, \\
&V'' + p_{21}U' + p_{22}V' + (q_{21} + \tfrac{1}{2}u_{21})\,U \\
&\qquad\qquad + [q_{22} + \tfrac{1}{4}(u_{22} - u_{11})]\,V = 0.
\end{aligned} \tag{3}
$$

The relation between (1) and (3) is a reciprocal one. The two systems have the same invariants. Their monodromic and transformation groups stand in a simple relation to each other. They are equivalent if, and only if, the integrating ruled surface of (1) is of the second order.

It is well known that the Riemannian geometry, assuming space to possess constant positive curvature, postulates the finiteness of the plane and the straight line and leaves room for only two assumptions concerning two straight lines. The first is that two straight lines always meet in two points (the antipodal points), at a distance πk from each other ($k \equiv$ constant of curvature); the second makes the plane a unilateral surface and postulates that the two antipodal points coincide in one point on opposite sides of the plane. The first assumption corresponds to the so-called spherical space, in which the geometry for two dimensions coincides completely with the spherical geometry in euclidean space and the geometry for three dimensions, according to Beltrami, necessarily presupposes a space of four dimensions, in which the spherical three dimensional space is the analogon of a sphere in three dimensional euclidean space. The second assumption belongs to the elliptic geometry proper, as expounded by Clifford, Klein, and Newcomb. The complete length of a straight line is πk, and we can pass from the part of the plane on one side of the straight line to the other, without crossing the line itself, by moving in a direction parallel to it. All the curves parallel to a straight line, that is, whose points are at equal perpendicular distances from the line, form a system of concentric circles, whose center is at a distance $\frac{1}{2}\pi k$ from the straight line. This distance is the greatest possible absolute distance and the straight line and the center are in relation of pole and polar. All rays at an angle $\frac{1}{2}\pi$ to the polar pass through the pole and all points at a distance $\frac{1}{2}\pi k$ from the pole lie on the polar.

It is this last form of the Riemannian geometry which Dr. Rabinovitch, in his paper, finds to involve internal contradictions. The procedure followed is both synthetic and analytic. The first considers two concentric small sectors, between the same two rays, of lengths r for one and $\pi k - r$ for the other sector ($r < \frac{1}{2}\pi k$), the radii being measured upon the same side of the plane. He shows that, as a direct consequence of the above postulates of the elliptic geometry, when the figure, as a rigid whole, is rotated about the pole through an angle π, the two sectors, measured on the same side of the plane as originally, interchange in magnitude, the larger taking up the position of the trace of the smaller and *vice versa*. The only escape from this contradiction is that the two antipodal points must be distinct, which was originally Beltrami's contention.

Further, a comparison of two formulæ from spherical geometry, which must hold for all surfaces of constant positive curvature, shows that the assumption of the period πk for a geodesic leads to the equation $+1 = -1$.

The paper intimates the possibility of a concrete and true interpretation of the analytical formulæ of the elliptic geometry in a space whose element is a line or some other geometrical figure instead of a point, with corresponding alteration of meaning of metrical terms, like angle, distance, etc.

The simplest type of problem in the calculus of variations,

$$I = \int F(x, y, y')\, dx = \text{minimum},$$

leads to the differential equation

$$y'' F_{y'y'} + y' F_{y'y} + F_{y'x} - F_y = 0.$$

The doubly infinite system of functions or curves satisfying this equation are called the extremals connected with the integral. The question considered in Dr. Kasner's paper,* stated for this simplest type of problem, is an inverse one: Given an arbitrary doubly infinite system of curves, is it always possible to regard it as the system of extremals connected with an integral I? This question is answered in the affirmative: there is an infinite number of integrals I, *i. e.*, of integrands F, which have the assigned system of curves for extremals. The determination of F depends upon

* After this paper had been presented, my attention was called by Professor Osgood to the dissertation of G. Hamel, Göttingen, 1901, from which I learn that the inverse problem of the calculus of variations has been discussed, and most of my results anticipated, by Hirsch, *Math. Annalen*, 1897, and Boehm, *Crelle*, 1900. E. KASNER.

the solution of a certain partial differential equation of the second order which is reduced by quadratures to a linear partial differential equation of the first order.

The more complicated types of problems in the calculus of variations are examined in a similar manner. But in no type except the simplest are the differential equations which present themselves as necessary conditions of general character, *i. e.*, the system of extremals is no longer an arbitrary system of curves or surfaces (in space of any number of dimensions). Thus an integral of type

$$\int F(x, y, y', y'')\, dx$$

leads to a four parameter system of extremals; but only a restricted class of four parameter systems of curves can be obtained in this way. The characterization of this class depends upon the consistency of a certain set of partial differential equations.

The results obtained are of interest in connection with Hilbert's recent investigations, as serving to characterize the types of differential equations which can be studied by means of corresponding problems in the calculus of variations. All ordinary equations of the second order, but only a restricted class of linear partial differential equations of the second order, can be so treated.

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MODERN METHODS OF TREATING DYNAMICAL PROBLEMS AND IN PARTICULAR THE PROBLEM OF THREE BODIES.

PRÉCIS OF FOUR LECTURES DELIVERED BEFORE THE ITHACA COLLOQUIUM, AUGUST 21–24, 1901.

BY PROFESSOR E. W. BROWN.

IN this course an attempt was made to give some idea of the methods which have been used during the last twenty-five years to obtain information about certain classes of dynamical problems and in particular about those problems which are ordinarily considered by physicists to be insoluble in all their generality. One of the most important of these problems and one specially considered in the lectures is the *Problem of Three Bodies*, which at

present can only be attacked in special cases. Attention was mainly directed to the *methods*—initiated by G. W. Hill, H. Gyldén, H. Poincaré, and others—and in order that these might be set forth, proofs were generally omitted, except when they were necessary for purposes of illustration; thus the lectures became descriptive to a large extent. Two of the objects aimed at were to show how the methods of modern analysis have been applied with some success to dynamical problems and to indicate the general lines on which research has been and can be carried on. It was desired to make the course interesting to an audience mainly composed of pure mathematicians. As no special knowledge on the part of the audience was assumed, it was necessary to devote most of the first lecture to the general principles of dynamics. The other topics selected were the existence of integrals, invariant integrals, divergent series, periodic solutions in general and those of the first class, characteristic exponents, periodic solutions of the second class and of the second species, asymptotic solutions and stability. There were four sessions, each of nearly two hours, with a ten minute interval in the middle.

1. *General Principles of Dynamics.**—The laws of motion and the energy equation were briefly touched on. The equations of dynamics can be put into a canonical form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i = 1, 2, \cdots, n),$$

where x_i, y_i are coördinates defining the position and motion of the system and F is a function of them. The transformations (tangential) which retain the canonical form when the defining coördinates are altered, can be made by means of a certain function S; this function has important properties when the Jacobian method is used to solve the equations by continued approximation, the approximations being carried out by varying the arbitrary constants. The types of transformations most frequently used are the *linear* and the *polar*, the latter being of the form

$$x' = \sqrt{2x}\cos y, \qquad y' = \sqrt{2x}\sin y.$$

In general, we can expand F in a convergent series

* See the astronomical treatises of Tisserand, Dziobek, Poincaré, Brown and many works on theoretical mechanics.

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots \equiv F_0 + R,$$

where μ is a given constant. The treatment consists in obtaining a general solution with $F = F_0$ and then in considering the arbitrary constants as variable when $F = F_0 + R$; in the Jacobian method the variable arbitraries satisfy a set of canonical equations. By first omitting $\mu^2, \mu^3, \cdots$ and then following out the same process, we can step by step approximate *formally* to the required solution. The application to astronomical problems consists in supposing the body under consideration to describe an ellipse in the first approximation ($F = F_0$); the arbitraries are then functions of the elements of this ellipse. Delaunay's lunar theory and the usual method of treating the planetary theory are examples.

Poincaré has frequently used the Jacobian method in a different way.* The solution, in fact, depends on the discovery of a solution of the partial differential equation

$$F\left(\frac{\partial S}{\partial y_i}, y_i\right) = C,$$

where S is such that $x_i = \partial S/\partial y_i$, and $F \equiv F(x_i, y_i) = C$ is an integral of the canonical equations. Instead of putting $F = F_0$ and finding the corresponding value of S, he puts

$$S = S_0 + \mu S_1 + \cdots, \qquad F = F_0 + \mu F_1 + \cdots,$$

and, expanding in powers of μ, determines $S_0, S_1, \cdots$ successively. Various difficulties arise. In astronomical applications S is not generally expansible in powers of μ but in powers of $\sqrt{\mu}$, just as in the inversion of $y = a_1 x + a_2 x^2 + \cdots$, when $a_1 = 0$, x is expansible in powers of $\sqrt{y}$. Again we usually require the solution to consist of periodic terms only, and special ways of defining the arbitrary constants must be used to effect this.

An equation frequently arising is

$$\frac{d^2x}{dt^2} + x(1 + \mu q_1 \cos t + \mu^2 q_2 \cos 2t + \cdots) = 0.$$

This equation has been treated by many writers.†

2. *Integrals.*—An integral of the equations

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \cdots, n)$$

* Les nouvelles méthodes de la mécanique céleste, chaps. IX, XI, etc.
† For references see Tisserand, Méc. cél., vol. 3, chaps. I, II.

is defined to be a function of the x_i such that when it is differentiated and the values of the dx_i/dt substituted, the result is identically zero. When we consider integrals which are one-valued analytic functions of the x_i, ten are known in the problem of three bodies; Bruns,* Poincaré,† and Painlévé ‡ have considered the question of the existence of other integrals and the answer is always in the negative.

3. *Integral Invariants.*§ Suppose that we have a p-tuple integral

$$I = \int f(x_1, x_2, \cdots, x_n; dx_1, dx_2, \cdots, dx_n) \equiv \int f(x_i, dx_i),$$

where f is algebraic and homogeneous of order p with respect to the dx_i, and the x_i satisfy the equations $dx_i/dt = X_i$, the limits of integration being over a certain range of values of the x_i. Let the equations be integrated so that the x_i are expressed in terms of t, the arbitraries being the values of the x_i for $t = 0$. Substituting, we obtain

$$I = \int \psi\,(t, a_i, da_i).$$

Take a range R_0 of possible values of the a_i and integrate along this range for $t = 0$; we obtain an integral

$$I_0 = \int_{R_0} \psi\,(0, a_i, da_i).$$

To each value of a_i in R_0 will correspond a set of values of x_i at time t; thus to the range R_0 will correspond a range R at time t giving

$$I = \int_R \psi\,(t, a_i, da_i).$$

If $I = I_0$ for all values of t, then I is an integral invariant.

An example is furnished by the motion of an incompressible fluid; the constant volume of any portion gives an integral invariant. The "circulation" in vortex motion furnishes another.

A *relative* invariant is one only existent for closed folds —using the language of n-dimensional geometry; an *absolute* invariant is one existent for open folds. A relative invariant of order p can be transformed into an absolute invariant of order $p + 1$ when the function under the integral sign is also a polynomial in the dx_i, in a manner analogous to

* *Acta Math.*, vol. 11.
† *Loc. cit.*, chap. V.
‡ *Comptes rendus*, vol. 130, p. 1699.
§ Poincaré, *l. c.*, chap. XXII.

Stokes's theorem for transforming from a line to a surface integral.

To an integral invariant of the equations $dx_i/dt = X_i$ corresponds an integral of the equations of variations (see section 7 below), and *vice versa*. As the equations of variations of the canonical equations of dynamics possess at least one integral, an integral invariant always exists for the equations of dynamics.*

One of the chief uses of integral invariants is as a means of verifying the results when the problem has been solved. If we confine ourselves to special forms of integral invariants, the maximum number possible in the three bodies problem can be stated and some of them are actually known. The unknown ones may not exist at all. Poincaré suggests that perhaps the method of Bruns, referred to above, might be used to furnish existence theorems.

5.† *Divergent Series.*‡—As many of the series which are used in astronomy are divergent, it is necessary to consider how the approximate values of the functions which they are supposed to represent may be obtained from them.

Suppose a divergent series

$$f_0 + \mu f_1 + \cdots$$

to represent formally a function φ, and let φ_p be the sum of the first $p+1$ terms of the series. If

$$\lim_{\mu=0} \frac{\varphi - \varphi_p}{\mu^p} = 0,$$

the series is said to represent φ asymptotically and the first p terms will give an approximation to φ, which will be the more exact the smaller μ is. The greatest value of p is generally the number of the smallest term and p is greater the smaller μ is.

A method analogous to Cauchy's proves that it is possible to use such series as solutions of the equations $dx_i/dt = X_i$, and it furnishes a limit to the error committed. The main point is that only a *limited degree of approximation* is possible; the practical effect in astronomy is to limit the time during which the series are available.

* Following Poincaré, we consider the equations of dynamics to be such that when put into a canonical form, they have the integral $F = C$, where F is independent of t explicitly.

† Owing to lack of time no. 4 (Least action) of the syllabus distributed at the meeting was omitted.

‡ Poincaré, *l. c.*, chaps. VIII, XIII; Borel, Leçons sur les séries divergentes, chap. I.

Another form of divergence arises with series like

$$A_0 + \Sigma_{m_1}\Sigma_{m_2} \frac{A_{m_1, m_2}}{m_1 n_1 + m_2 n_2} \cos (m_1 n_1 + m_2 n_2)t,$$

where m_1, m_2 are positive or negative integers, A is a non-vanishing coefficient and n_1, n_2 are any numbers. For convergence it is necessary that

$$\left| \frac{A}{m_1 n_1 + m_2 n_2} \right| < \varepsilon, \text{ a finite number,}$$

which cannot hold if n_1/n_2 is commensurable with unity. In applications n_1, n_2 are observed numbers and therefore determined to a limited number of places of decimals. It may be shown that we can generally find numbers n_1', n_2' as close as we wish to n_1, n_2 such that the series either diverges or converges as we desire; in the latter case the convergence is not uniform.

The final result for astronomical series is as follows:

The canonical equations

$$(1) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \quad (i = 1, \ldots, p),$$

where

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots \qquad \text{(convergent)},$$

are satisfied *formally* by

$$(2) \qquad \begin{aligned} x_i &= x_i^{(0)} + \mu x_i^{(1)} + \ldots, \\ y_i &= w_i + \mu y_i^{(1)} + \ldots, \end{aligned} \qquad w_i = n_i t + \tilde{\omega}_i,$$

where

$$(3) \quad x_i^{(k)} (\text{or } y_i^{(k)}) = A_0 + \Sigma \frac{B \cos (m_1 w_1 + \ldots + m_p w_p + h)}{m_1 n_1 + \ldots + m_p n_p};$$

the m_j are integers, n_i, $\tilde{\omega}_i$, h, A_0, B constants, and the sign of summation includes all terms the type of which is written down (the coefficient B is different for each term). By what has just been said *we may consider the series* (3) *as convergent, but the series* (2) *do not converge.* If (2) converged, it is shown that another integral of (1) would exist—contrary to the theorems of section 2 above. This does not necessarily imply instability; it may be only a defect of our analytical mode of representation.

6. *Periodic Solutions.*—A system is said to be in periodic motion when, after the lapse of a certain interval, its posi-

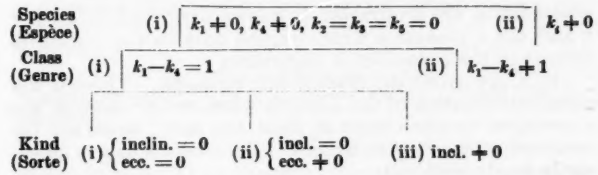
tion and motions are the same as at the beginning of the interval. For n bodies the definition is extended to their relative positions and motions. There is no general method of finding all the periodic solutions of a given system, but existence theorems will deduce a range of them from a known one. There may be separate systems of periodic solutions; the method deals with only one system at a time.

Poincaré uses different kinds of periodic solutions which may be exhibited for the problem of three bodies as follows:

Let $y_1, y_2, y_3, y_4, y_5, y_6$ represent the six angles used in the motion of two small bodies about a third large one; y_1, y_4, the angular positions at time t; k_1, k_2, k_3, k_4, k_5 any positive or negative integers.

A periodic solution exists if $y_1 - y_6$, $y_2 - y_6$, $y_3 - y_6$, $y_4 - y_6$, $y_5 - y_6$ increase respectively by $2k_1\pi$, $2k_2\pi$, $2k_3\pi$, $2k_4\pi$, $2k_5\pi$ when t increases by a period.

The classification is shown by the following tree:



*Periodic solutions of the first class** are those in which the relative orbits are closed curves with one turn round the primary body. These include the orbits of Hill† and Darwin.‡

Poincaré's method is to suppose that a periodic solution exists for a particular value of μ and then to prove the existence for a range of values of μ. There may be several periodic solutions for a given μ. As μ is varied, they will appear and disappear in couples like the imaginary roots of an algebraic equation. The results differ according as, in the equations $dx_i/dt = X_i$, the X_i do or do not contain t explicitly, and according as the equations do or do not admit of an integral. The application to the orbits of Hill and Darwin is immediate, these being of the first *kind;* the second and third kinds, although they can exist, await applications.

* Poincaré, *l. c.*, chap. III.

† *Amer. Jour. Math.*, vol. 1.

‡ *Acta Math.*, vol. 21. See also S. S. Hough, *ibid.*, vol. 24; and Schwarzschild, *Astr. Nach.*, vol. 147.

7. *Characteristic Exponents.**—Suppose that the equations $dx_i/dt = X_i$ admit a known periodic solution

$$x_i = \phi_i(t),$$

and that we try to obtain a solution (not necessarily periodic)

$$x_i = \psi_i(t) + \xi_i,$$

not very different from the former. We substitute the latter in the differential equations and, omitting squares and products of the ξ_i, obtain the *equations of variations*

$$\frac{d\xi_i}{dt} = \xi_1 \frac{\partial X_i}{\partial x_1} + \xi_2 \frac{\partial X_i}{\partial x_2} + \cdots + \xi_n \frac{\partial X_i}{\partial x_n}.$$

The general solution of these is

$$\xi_i = A_1 e^{\alpha_1 t} S_{i1} + \cdots + A_n e^{\alpha_n t} S_{in},$$

where the A_i are arbitraries, S_{ij} known periodic functions of t, and the α_i constants which depend only on the constants present in the equations of variations.

The α_i are called the *characteristic exponents.* There is the usual modification of the solution when two or more of the α_i are equal or when some of them are zero: terms are introduced with $t, t^2, \cdots$ as factors. These terms do not necessarily imply instability.

The discussion turns on (i) whether the α_i are real, imaginary, or complex; (ii) whether one or more of them is zero; (iii) whether two or more of them are equal. The three cases depend mainly on the presence of t explicitly in the X_i, and on the existence of an integral of the original equations. The equations of dynamics (see the first footnote to p. 107) give two exponents zero and the others equal and opposite in pairs. In most problems of celestial mechanics the α_i are expanded in powers of $\sqrt{\mu}$.

8. *Periodic solutions of the second class* † give more than one revolution round the primary before the curve is reëntering. The only method at present available is to investigate the existence of such solutions near a solution of the first class. Consider one solution of the equations of variations

$$\xi_i = Ae^{\alpha t\sqrt{-1}} S,$$

* Poincaré, *l. c.*, chap. IV.
† Poincaré, *l. c.*, chaps. XXVIII., XXXI.

and let T be the period of the first-class solution and therefore of S, and suppose α to be real. The ξ_i are then expressible by sines and cosines of the angles αt and of multiples of $2\pi t/T$. If $\alpha = 2m/kT$, m, k being integers, then after k revolutions the ξ_i take up again their old values and the period is kT. The difficulties to be dealt with arise from the fact that only the first powers of the ξ_i are at first included. Special applications are made to the Darwin orbits, and a connection between the conditions for the existence of a second-class orbit and the passage from stability to instability of the Darwin orbits is shown.

9. *Periodic Solutions of the Second Species.*—Poincaré* conceives a way in which these may exist. He supposes that two very small bodies are describing orbits round a large one and that the orbits are so placed that a collision very nearly takes place at definite intervals. The orbits are ellipses whose elements remain practically constant except near each collision point where they are suddenly and greatly changed. Thus all the elements may vary and the motion may be periodic. It may be possible by methods of continuity to deduce periodic solutions in which collisions will not nearly occur; but such orbits would probably be very unstable.

10. *Asymptotic Solutions.*†—Suppose that we have a problem of dynamics with two degrees of liberty. Using the periodic solution, and then a near solution, we obtain results of the form

$$\begin{matrix} x_i \\ y_i \end{matrix} = \varphi_i(t) + \text{series in powers of } Ae^{\alpha t},\ A'e^{-\alpha t} \quad (i = 1, 2),$$

where $\varphi_i(t)$ and the coefficients of the series are periodic. Suppose α real and positive; then A, A' are real. If $A = 0$, the solution approximates to $\varphi_i(t)$ for t very great, and the orbit, which starts with a slight modification of the periodic solution, soon differs much from it, but as time goes on approaches it more and more nearly. The solution is called *asymptotic*. A second kind of asymptotic solution is obtained with $A' = 0$, $A \neq 0$.

We may discuss the problem by considering x_1, x_2, y_1 as coördinates of a point in space; y_2 is determined from the energy equation $F(x_1, x_2, y_1, y_2) = C$, where C is a *given* constant. Then if t, $A'e^{-\alpha t}$ be treated as parameters, the point (x_1, x_2, y_1) always lies on a surface, called *asymptotic*. Simi-

* *Loc. cit.*, chap. XXXII.

† Poincaré, *l. c.*, chaps. VII, XXXIII.

larly we have a second asymptotic surface for $A' = 0$, $A \neq 0$. The representation may be changed by transformation of coördinates if necessary for convenience of discussion. The periodic solution ($A = 0$, $A' = 0$) lies on both surfaces and therefore on their intersection. The particular case of the three bodies problem in which the third body is very small (of "zero" mass) and the other two describe circular orbits about one another can be treated. With properly chosen coördinates instead of x_1, x_2, y_1, it is shown that two curves on an asymptotic surface which correspond to trajectories never cut one another, and that two similar curves on different asymptotic surfaces may cut.

The doubly asymptotic solution is one which approaches a periodic solution when $t = +\infty$ and also when $t = -\infty$. The periodic solutions thus approached may be the same (homoclinic) or different (heteroclinic). A homoclinic orbit never cuts itself. There are two kinds of homoclinic solutions and an infinite number of each kind such that a trajectory never cuts one of its own kind but cuts one of the other kind an infinite number of times. Very little is known of heteroclinic solutions.

11. *Stability.*—In the problem of n bodies the *sufficient* conditions for stability appear to be:

1°. That the bodies never get infinitely distant from one another;

2°. That their mutual distances never descend below a certain limit;

3°. That each body passes an infinite number of times as near as we wish to any point through which it has once passed;

4°. That a small external disturbance shall not affect the fulfilment of conditions 1°, 2°, 3°.

It is not known whether these are all *necessary*. In the general problem we do not know whether any one of them is satisfied. Poincaré states the first three conditions* and proceeds to investigate the third in some detail. He shows that when the bodies are started so as always to remain within a closed surface, those that satisfy 3° are the rule and the others are the exception, the former being infinite in number compared with the latter. The proof depends on the existence of an integral invariant. In a particular case of the sun-earth-moon system the numerical limits for 1°, 2° have been found.†

* *Loc. cit.*, chap. XXVI.

† Hans Haffel, Inaugural Dissertation, Göttingen, 1900.

Literature.

Full references to the literature of celestial mechanics are given in the following works:

Gautier: "Essai historique sur le problème des trois corps." Paris, 1817.

Cayley: "On the progress of the solution of certain problems in dynamics." B. A. Reports, 1862.

Dziobek: Die mathematischen Theorien der Planeten-Bewegungen. Leipzig, 1888.

Tisserand: Traité de mécanique céleste. Paris, vol. 1, 1889; vol. 2, 1891; vol. 3, 1894; vol. 4, 1896.

Brown: Introductory treatise on the lunar theory. Cambridge, 1896.

Whittaker: "Report on the progress of the solution of the problem of three bodies. B. A. Reports, 1899.

Wislicenus: Astronomischer Jahresbericht. Berlin, vol. 1 for 1899, vol. 2 for 1900.

It may be added that most of Gyldén's work is to be found in the volumes of the *Acta Mathematica* and in the "Traité analytique des orbites absolues des huit planètes principales" (Stockholm, vol. 1, 1893).

The treatise of Poincaré, frequently referred to above, has been mainly used in the preparation of these lectures. It consists of three volumes published in 1892, 1893, 1899, respectively, and is an expansion of the memoir in vol. 13 (1890) of the *Acta Mathematica* entitled, "Sur le problème des trois corps et les équations de la dynamique." Most of this author's other papers on these subjects will be found in the volumes of the *Bulletin Astronomique.*

THE HAMBURG MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG, SEPTEMBER, 1901.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held at Hamburg, September 22–28. In the absence of the president of the society, Professor von Dyck, Professor Hilbert occupied the chair. At the business meeting Professor Gutzmer reported on a plan for changing the *Jahresbericht* into a monthly periodical to be the official organ of the society, and to contain mathematical papers, univer-

sity and personal news, accounts of the meetings of other mathematical societies, and similar items of general interest. The plan was adopted and Professor Gutzmer elected editor. The financial management will be conducted by the Teubner publishing firm.

The following papers were read at the meeting:

Professor SCHOUTE, Groningen: "On the mobility of a nullsystem N_{2n-1} in R_{2n-1}."

Professor SCHUBERT, Hamburg: "On the number of constants of the n-dimensional generalization of polyedra."

Dr. JAHNKE, Berlin: "On rotations in four dimensional space."

Professor STUDY, Greifswald: "A new branch of geometry."

Professors KLEIN, Göttingen; MEYER, Koenigsberg; SOMMERFELD, Aachen: "An account of the present condition of the Encyklopädie der mathematischen Wissenschaften."

Professor VON LILIENTHAL, Münster: "The geometry of motion in its application to differential geometry."

Professor STÄCKEL, Kiel: "Arithmetical characteristics of analytical functions."

Professor EBERHARD, Halle: "Contribution to the theory of equations."

Professor STÄCKEL, Kiel: "Report on the development of instruction in applied mathematics in the German universities."

Professor ENGEL, Leipsic: "Higher differential quotients."

Professor MEYER, Koenigsberg: "The extension of the theorems of Henrici and Ivory."

Professor MITTAG-LEFFLER, Stockholm: "A criterion for finding the singularities of analytical functions."

Herr TORKA, Berlin: "On a new geometry of motion."

Professor SCHILLING, Göttingen: "New kinematical models in tooth gear and their relation to the theory of contact transformations."

Professor HAUK, Charlottenburg: "The relations between three parallel projections of a space system."

Professor ADLER, Prague: "Spherical representation of surfaces and its application in descriptive geometry."

Professor MÜLLER, Königsberg: "On the analogue to Lie's spherical geometry in the domain of the straight line."

Professor LONDON, Breslau: "On a particular kind of converging point series."

Professor HILBERT, Göttingen: "On certain recent mathematical dissertations."

Dr. ZERMELO, Göttingen: "On the theory of the shortest lines."

Papers on astronomy and geodesy were read by Professor CHARLIER, Lund; Dr. HALM, Edinburgh; Dr. HARTWIG, Hamburg; Dr. MARCUSE, Berlin; Dr. EBERT, Kiel.

In Professor Schoute's paper the nullsystem was developed analytically and geometrically for higher space (necessarily of an odd number of dimensions). The motion under which the nullsystem remains invariant in space of $2n-1$ dimensions is the most general motion of a rigid body in that space—a translation and $n-1$ rotations. The relation of the nullsystem to mechanics was mentioned.

Professor Schubert showed that the number of constants of the n dimensional polyedra could not in general be expressed by means of the number of constituent elements of lower dimensions. The number of constants of the n dimensional tetrahedron, hexahedron, and octahedron was developed as $n(n+1)$, $2n^2$, and $2n^2$, respectively, for $n > 4$.

Dr. Jahnke expressed the rotations in four dimensional space by the parameters Euler used to represent the direction cosines of an orthogonal system and showed the relation existing between a four dimensional rotation and two correlated three dimensional rotations. The results find application to the motion of a body in an ideal liquid and to the rotation about its center of gravity of a body carrying an axis about which a second body rotates.

In Professor Study's paper a new line geometry was developed. In the ordinary line geometry the ∞^2 ideal (uneigentlich) or infinitely distant lines are adjoined to the ∞^4 actual (eigentlich) lines to form a closed continuum as defined by Cantor. This continuum can be represented continuously on a quadratic manifold in a plane space of five dimensions. The collineations are determinate, single valued, and continuous.

The manifold of the ∞^4 actual lines can, however, be amplified to a continuum in an entirely different way. Professor Study adjoins the ∞^2 infinitely distant *points* to the ∞^4 actual lines. This manifold forms a closed continuum which can be represented continuously on a manifold of sixth order in a plane space of eight dimensions. There exists a continuous group with 17 parameters whose transformations are determinate, single valued, and continu-

ous, and the invariant configuration of this group is that of an actual line with all other lines which cut it at right angles. These transformations define a new geometry which may be compared to the Plücker line geometry, but which differs radically from it, as was shown by several theorems which Professor Study stated.

Professors Klein, Meyer, and Sommerfeld reported on the progress of the "Encyklopädie der mathematischen Wissenschaften." The difficulties in its preparation are much greater than at first supposed, but they are being steadily overcome, and the work is progressing satisfactorily in all directions.

Professor von Lilienthal mentioned the twists studied by Beltrami in connection with curvature of curves, and then considered the infinitesimal displacement of a rectangular trihedron, one edge coinciding with the normal to a given surface at a point P, and the two others with tangents to two curves C_1 and C_2 of a given orthogonal family, as P moves on the surface. The intersections of the tangent plane to the surface at P with the axes of the linear family of twists which produce the displacement, lie on a straight line L which passes through the middle points of geodetic curvature of C_1 and C_2. Let the principal radii of curvature at P be ρ_1 and ρ_2, and the points of intersection of the line L with the principal tangents at P be P_2 and P_1 respectively. If $PP_1 = l_1$, $PP_2 = l_2$, and if we write K_1 and K_2 for the radii of geodetic curvature of C_1 and C_2, and φ for the angle between C_1 and the line of curvature corresponding to ρ_1, then

$$\frac{1}{K_1} = \frac{\cos\varphi}{l_2} - \frac{\sin\varphi}{l_1}, \qquad \frac{1}{K_2} = \frac{\cos\varphi}{l_2} + \frac{\sin\varphi}{l_1},$$

an expression for the geodetic curvature similar to that of Euler for the curvature of a normal section. Comparing these expressions with those of Liouville for K_1 and K_2 we have a geometrical condition for the derivatives of φ with respect to the arcs of the lines of curvature, namely,

$$\frac{1}{l_1} = \frac{1}{R_1} + \frac{d\varphi}{ds_1}, \qquad \frac{1}{l_2} = \frac{1}{R_2} - \frac{d\varphi}{ds_2},$$

where R_1 and R_2 are the radii of geodetic curvature of the lines of curvature. It was shown that the axes of the

above mentioned twists form a cylindroid, the properties of which were considered.

Professor Stäckel's paper was an account of his investigations in continuation of those published by him in the *Annalen* and in the *Comptes rendus*. In attempting to characterize the algebraic functions arithmetically, Professor Stäckel finds that an analytical function $p(x, y)$ is not necessarily algebraic when it is assumed that, in the equation $p(x, y) = 0$, to every algebraic x there corresponds an algebraic y, and to every algebraic y there corresponds an algebraic x. An arithmetical criterion for algebraic functions has not yet been found. The following theorem was proved: When for a single valued analytical function, having its one essential singularity at infinity, and possessing an algebraic addition theorem, such a point exists that the coefficients in the development of the function in a power series about the point are all algebraic numbers, then the function can be simply periodic only when the period is a transcendental number. In conclusion Professor Stäckel investigated the nature of the periods of such elliptic functions as admit of complex multiplication.

The object of Professor Eberhard's paper was to find the number and approximate the position of the real roots of an algebraic equation with real coefficients. Let this equation, of degree n, be $f(x) = 0$. Professor Eberhard builds two auxiliary equations of degree at most $n - 1$, namely,

$$\frac{d}{dx} f(x) = 0, \qquad \left[\frac{d}{dx'} f\left(\frac{1}{x'}\right)\right]_{x'=\frac{1}{x}} = 0,$$

i. e., the derived equation, and the reciprocal of the derived equation of the reciprocal. If the real roots α_i, β_i of these two equations be arranged in one series according to magnitude, then the position of the real roots of $f(x) = 0$ can be found relative to the terms of this series, *i. e.*, relative to the roots α_i, β_i, of equations of lower degree.

Professor Stäckel gave, in his second paper, an account of the development in instruction in applied mathematics at the German universities. With the founding of technical schools the study of applied mathematics at the universities lost strength, and only in the last few years, and largely through the efforts of Professor Klein, has the work begun to assume a place of importance. The universities of Göt-

tingen and Strassburg were the first to offer regular and extended courses in descriptive geometry, geodesy, and technical mechanics, and their lead is being rapidly followed by the other institutions.

The object of Professor Engel's paper was the generalization of the Clebsch "Konnexcoördinaten" for the geometry of differential equations of order higher than the first. In the *Leipziger Berichte*, 1893, Professor Engel showed how this could be accomplished, but the complicated form of the result prevented its practical application. Professor Study (*Leipziger Berichte*, 1901) has solved the problem for a differential equation of second order in the plane, and Professor Engel gave in the present paper the corresponding development for partial differential equations of second order in space.

Professor Meyer's paper dealt with the generalization of the theorems of Henrici and Ivory. If P_1, P_2 be two points on a quadric surface, F_1, and if these points go into P_1', P_2' by the transformation which carries F_1 into any confocal quadric F_2, then in general (Ivory's theorem) $\overline{P_1P_2'} = \overline{P_2P_1'}$; and if P_1 and P_2 lie on a generating line of F_1, then (Henrici's theorem) $P_1P_2 = P_1'P_2'$. Professor Meyer considered the algebraic identities which express these theorems and proved that each theorem was characteristic for confocal quadrics. If the ratio of the distances be assumed to be any constant, then the quadrics F_1 and F_2 must be similar and similarly placed. A generalization to pairs of coaxial quadrics was also made.

The generalization to surfaces of higher order which have a conic at infinity is especially interesting. Consider a surface of fourth order which has the conic C_2 at infinity. If O be any point in space, and O_1, O_2, O_3 the principal axes of the cone from O to C_2, then there exists an ∞^1 family of affine transformations A with the principal axes O_1, O_2, O_3, of such a nature that the distance between any two points P_1, P_2 of the surface of fourth order, whose joining line meets the surface at infinity, is invariant with respect to the transformations A. Similar theorems hold for surfaces of order n.

The position of the theorems of Henrici and Ivory in the theory of quadric surfaces is made more clear by the following considerations. Let the points P_1, P_2 have *any* position in space. The distance P_1P_2 remains invariant under a six parameter group of collineations. If however P_1, P_2

be restricted to lie on a generating line of a *fixed* quadric, then the six parameter group is extended to a seven parameter group by the addition of the above mentioned ∞^1 family of affine transformations A. The case of Ivory's theorem is similar; the collineations which change *every* line P_1P_2 into $P_1'P_2'$ in such a way that $\overline{P_1P_2'} = \overline{P_2P_1'}$ form a two parameter family S of "symmetric rotations." If P_1, P_2 lie on a fixed quadric, then the ∞^1 family of affine transformations A is added to S to form a three parameter group of collineations under which $\overline{P_1P_2'} = \overline{P_2P_1'}$.

Professor Mittag-Leffler spoke on the singularities of analytical functions. Hadamard and Lindelöf have given criteria for single points on the circle of convergence of a power series. The problem may be stated in another way and the analytical continuation of a power series be demanded along a given line. Professor Mittag-Leffler finds how far this is possible by building the necessary and sufficient condition for a singular point on the given line.

Herr Torka's paper dealt with the geometry of motion produced by various links and linkages.

Professor Schilling explained his new and interesting kinematical models for the generation of curves and pointed out their relation to Lie's contact transformations. The paper was illustrated by the working models and by projected diagrams.

Professor Hauk gave the conditions under which projective systems in three planes can be the parallel projections of a single space system, derived the relations which then exist between the elements of this system, and gave various methods of determining it.

Professor Adler's paper dealt with the spherical representation of surfaces. The polar of a given surface with respect to a unit sphere is projected from the center to the surface of the sphere. The lines of curvature of the original surface go into confocal spherical conics. The spherical representation of the lines of equal illumination of any surface in parallel light are also the lines of equal illumination of the sphere in the same light, and on this theorem rests the application to descriptive geometry.

Professor Müller showed that the projective geometry

on a straight line forms an analogue to Lie's spherical geometry.

Professor London gave a construction which approximates to the point of convergence of certain point series (Punktfolgen) to any desired degree of precision. Let the points of a line be projectively related. To any initial point P_1 construct the corresponding point P_2, to P_2 again the corresponding point, etc. If the projective relation has real double points and does not form an involution, then the point series formed in the above manner converges to that double point which is furthest from the vanishing point. The point series formed similarly from the inverse projective relation converges to the other double point. If the double points are imaginary the points of the series lie everywhere dense on the line. Similar relations hold in the case of collineations of a plane, and also for three and higher dimensions.

Professor Hilbert spoke on the theses which have been written under his direction at Göttingen during the last two years. These may be divided roughly into three groups: the theses of Reid (1899), Hilbert (1900), Rueckel (1901), Bernstein (1901), and Fräulein v. Saposki (not yet appeared), deal with theory of numbers; those of Hedrick (1901), Noble (1901), J. Müller, and Fräulein v. Gernet (not yet appeared), with the calculus of variations and differential equations, and those of Feldblum (1899), Dehn (1901), Boy (1901), Zoll (Göttingen Preisarbeit, 1901), and Hamel (1901), with geometrical questions. Professor Hilbert spoke in detail on the last group only, and especially on the last three theses.

Herr Boy has succeeded in finding a finite closed surface without singularities whose points have a one to one correspondence to the points of the projective plane, and which has, therefore, the same relation to the projective plane as the sphere to the complex plane. The surface cuts itself but has no pinch points. It is, in a way, the simplest surface next to the sphere, as it has one maximum, one minimum and one saddle point.

Herr Zoll has been able to find closed surfaces without singularities upon which all geodetic lines are closed.

Herr Hamel has studied the geometries in which it is not assumed that the angles at the base of an isosceles triangle are equal, but instead that the straight line is the shortest distance. It is shown that there exist geometries of this

nature other than the two already found by Minkowski and Hilbert.

Dr. Zermelo's paper treated of three problems in the theory of geodetic lines: the absolute shortest line; the shortest line in a bounded region; the shortest line with restricted steepness.

A line on a surface is an absolute or relative shortest line, according as it is the shortest of all possible lines on the surface between two given points, or the shortest of all lines that lie in a strip as narrow as we please about it. Consider a point 0, and an extremal (a line satisfying Lagrange's equation) L through 0. It is known that the extremal L is the relative shortest line up to its point of tangency 1, with the envelope of extremals through 0; *it ceases, however, to be the absolute shortest line before the point* 1 *is reached.* An absolute shortest line l, $l_1 < L_1$, (where the subscript letters denote lengths) must exist* between the points 0 and 1 and must be an extremal through 0 which *cuts* the extremal L and the envelope at the point 1. If the parameter in the equation of l be so varied that the point of intersection a of l with L moves from 1 towards 0, then, from reasons of continuity, we have $la < La$, up to a certain point 2 where $l_2 = L_2$. The locus of such points 2 for all extremals through 0 forms a curve which Dr. Fermelo calls the "Doppelabstandscurve." This curve passes in general through cusps of the envelope.

The shortest line in a bounded plane region consists of straight lines and portions of the boundary convex towards the interior. The following theorem was given: In any simply connected plane region, the shortest line between two given points is uniquely determined, and is therefore the absolute shortest.

The third problem is that of shortest lines with restricted steepness, $K = \frac{dz}{ds} < K_0$, on the surface $z = \varphi(x, y)$. The shortest lines consist of geodetic lines and lines of constant steepness, $K = K_0$. The following theorem was proved: Every curve of greater steepness, $K > K_0$, can be replaced by zigzag lines of constant steepness, $K = K_0$, which run as near as we please to the curve, and which satisfy the conditions of the variation problem.

I wish to thank the speakers who so kindly responded to my requests for information, and to acknowledge the use

* Proved by Hilbert. See Noble, Diss., Göttingen, 1901.

which the notes of Dr. Schröder, secretary of the meeting, have been to me.

C. M. MASON.

GÖTTINGEN,
October, 1901.

SOME CURIOUS PROPERTIES OF CONICS TOUCHING THE LINE INFINITY AT ONE OF THE CIRCULAR POINTS.

BY E. V. HUNTINGTON AND J. K. WHITTEMORE.

(Read before the American Mathematical Society, October 26, 1901.)

THE subject of the present note was suggested by the apparent contradiction between the following well known theorems of modern analytic geometry:

1. Through any four given points of the plane there pass two parabolas.

2. Through the vertices and the orthocenter of a triangle there pass no conics but equilateral hyperbolas.

The difficulty here presented may be removed as follows: We know that conics through four points determine an involution on any line of the plane. Since the conics of theorem 2 are equilateral hyperbolas the involution which they determine on the line infinity will have the circular points I and J as double points. Two of the hyperbolas are therefore tangent to the line infinity, one at I and the other at J. But any conic tangent to the line infinity is a parabola; therefore these two hyperbolas are parabolas, and the contradiction with Theorem 1 disappears.

We shall now investigate these curves more closely.

Let S be any conic touching the line infinity at one of the circular points, I.

The conic S is a parabola, being tangent to the line infinity. But the straight lines joining any finite point with the infinite points of the curve are perpendicular, since any line through I is perpendicular to itself; hence S is also an equilateral hyperbola.

The center of S, *i. e.*, the pole of the line infinity, is the point I.

The four foci, *i. e.*, the points of intersection of tangents from I and J, coincide with J.

The four directrices coincide with the polar of J, and pass therefore through I.

The eccentricity m/n is wholly indeterminate ;* for while the distance n of any point of the conic from the directrix is in general infinite (since the directrix is a line through I), the distance m from the focus is always indeterminate (since the focus is at J).

The two perpendicular axes, joining the center with the foci, coincide with the line infinity, which, like every line through I or J, is perpendicular to itself.

The four vertices, where the conic intersects the axes, coincide with I.

The asymptotes, being tangents to the conic drawn from its center, coincide with the line infinity.

Any conic S' touching the line infinity at the other circular point J will of course have corresponding properties, obtained by interchanging I and J.

Through any three finite points A, B, C of the plane there pass one conic S and one conic S'.

For the three given points, and a fourth point (I or J) with its tangent, determine a conic.

The two conics S, S' through A, B, C intersect also in the orthocenter, O, of the triangle ABC.

For, let A, B, C have the coördinates $(a, 0)$, $(b, 0)$, $(0, 1)$, whence O will be $(0, -ab)$; and let I and J be the infinitely distant points on the lines $x + yi = 0$ and $x - yi = 0$ respectively.

Then the two parabolas S and S' through A, B, C and I or J will have the equations

$$x^2 \pm 2xyi - y^2 - (a + b)x + (1 - ab)y + ab = 0,$$

where the upper sign goes with S and the lower with S'; and both of these curves contain the point O.

The directrices of the two parabolas S, S' through A, B, C and O intersect in the center of gravity G of these four points.

For the directrices of S and S', being the polars of J and I respectively, are found to have the equations

$$4x \pm 4yi = (a + b) \pm (1 - ab)i,$$

and both of these lines evidently contain the real point $\left(\frac{a+b}{4}, \frac{1-ab}{4}\right)$, which is the center of gravity G of the four given points.

* This is consistent with the fact that the eccentricity of a parabola is 1, and that of an equilateral hyperbola $\sqrt{2}$.

Further: *the directrices of S and S' are the asymptotes of the nine-points circle of the triangle ABC.*

For, the nine-points circle (passing through the middle points of the sides) has the equation

$$2x^2 + 2y^2 - (a + b)x - (1 - ab)y = 0;$$

hence its center is the point G already found, and the directrices, joining G with I and J, are the asymptotes of the circle.

The barycentric equation of the conic S referred to the triangle ABC which it circumscribes is

$$\beta\gamma(z_2 - z_3)^2 + \gamma\alpha(z_3 - z_1)^2 + \alpha\beta(z_1 - z_2)^2 = 0, \qquad (1)$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the rectangular coördinates of the points A, B, C, and $z = x + yi$.

This equation may be found by expressing the fact that the tangent to S at the point I (whose barycentric coördinates are $\alpha : \beta : \gamma = z_2 - z_3 : z_3 - z_1 : z_1 - z_2$) coincides with the line infinity (whose barycentric equation is $\alpha + \beta + \gamma = 0$).

The equation of the conic S' through the same three points is

$$\beta\gamma(\bar{z}_2 - \bar{z}_3)^2 + \gamma\alpha(\bar{z}_3 - \bar{z}_1)^2 + \alpha\beta(\bar{z}_1 - \bar{z}_2)^2 = 0, \qquad (2)$$

where $\bar{z} = x - yi$.

When the triangle of reference ABC is *equilateral*, these equations assume the simple forms

$$\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta = 0, \qquad (1')$$

$$\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta = 0, \qquad (2')$$

where ω is one of the imaginary cube roots of unity.

PICARD'S TRAITÉ D'ANALYSE.

Traité d'Analyse. Par ÉMILE PICARD. Deuxième édition revue et corrigée. Tome I. Paris, Gauthier-Villars, 1901. 8vo, xvi + 483 pp.

IN looking back over the last half of the nineteenth century, it is interesting to notice how slowly the great ideas in analysis which we associate with the names of Riemann

and Weierstrass gained general currency in the mathematical world. Thirty years ago these ideas were only just beginning to be really understood by a small circle of mathematicians in Germany. Ten years later the great mathematical revival in France gave evidence that the same had become true in that country. Even ten years ago, however, the number of persons to whom these ideas were familiar in their various aspects was still small, and it was hardly possible to become initiated into them except through personal contact with a master. That this is no longer the case is due to a number of excellent text-books which have appeared during the last ten years, and to none so much, perhaps, as to the Traité d'Analyse by M. Picard, with which it is to be hoped that every reader of the BULLETIN is familiar. The first volume of this treatise which was published in 1891 has been followed by two others;* and now before the fourth and final volume appears, we are pleasantly surprised by a new edition of volume one.

The changes which have been made in this new edition are chiefly in the nature of small additions at various points, the total increase in the size of the volume being only twenty-six pages. The number of bibliographical references scattered through the volume has been increased, and certain sections have been shifted to a new position; thus, for instance, Bonnet's second law of the mean (often attributed to DuBois-Reymond) has now received a place (page 9) alongside of the first law of the mean. Among the additions and changes, the more important are the following:

In the first chapter § III and § IV are entirely new. The first of them relates to integrable and non-integrable functions according to Riemann. In the second Jordan's necessary and sufficient condition that a plane curve be rectifiable is given without proof, and then follows an account of the Peano-Hilbert curve which fills a square.

In the first note on page 27 Pringsheim's necessary and sufficient condition that a function of a real variable be developable by Taylor's theorem is stated.

A different method from that used in the first edition has been adopted, on pages 71–76, for the reduction of abelian integrals of the second kind.

The formula for the area of a curved surface is reduced on page 123 from the form involving Jacobians to the form $\int\int \sqrt{EG - F^2}\, du\, dv$.

* A review of the first two volumes of the Traité, by the late Professor Thomas Craig, appeared in the BULLETIN for November, 1893 (1st ser., vol. 3, no. 2, pp. 39-65).

On pages 202–203 a paragraph on doublets has been added.

§ II of Chapter VIII is entirely new and treats of uniformly convergent series whose terms are discontinuous functions. Both here and in connection with the Peano-Hilbert curve it is shown how continuous functions can fail to have a derivative.

Mertens' proof of the theorem that two convergent series may be multiplied together term by term provided one of them is absolutely convergent is reproduced on page 224.

On pages 283–287 two additional proofs are given of Weierstrass's theorem that any function $f(x)$ continuous throughout the interval $\alpha \leqq x \leqq \beta$ can be represented by a series of polynomials absolutely and uniformly convergent throughout this interval. The first of these proofs is essentially Weierstrass's * and depends, as does Picard's, upon the possibility of making a divergent Fourier's series convergent by the introduction of suitable "convergence factors" † into the terms of the series. The second proof, due to Volterra, is extremely simple, but assumes the ordinary theory of Fourier's series.

In Chapter XII, paragraph 25, which in the first edition related only to twisted cubics, has been extended by two pages so as to cover twisted quartics as well.

On the last two pages of the volume a paragraph has been added concerning maps in which *areas* instead of angles remain unchanged.

The moderation and good judgment displayed by the author in making these additions is at once obvious and in fact I think it was these two qualities, moderation and good judgment, which, together with the proverbial lucidity of the French style, more than anything else, have made the Traité what it is—a classic. Few books have done so much as this one to make accessible those parts of mathematics in which no real progress can be made by one who has not a firm grasp of the rigorous methods of modern analysis, and yet the ε's and the δ's, with which most writers who strive for rigor surfeit their readers, are most sparingly used here. The intelligent reader of these volumes will learn what are the real difficulties to be overcome and what is the nature of the methods to be used, far better than he could have done if these salient points were obscured by a mass of detail in the form of inequalities.

* Or rather a special case, particularly emphasized by Weierstrass himself, of the very general treatment given by him.

† Cf. Sommerfeld, "Die willkürlichen Functionen in der mathematischen Physik." Dissertation, Königsberg, 1891.

The first two-thirds of the present volume forms essentially an introduction, though a somewhat unsystematic one, to the theory of functions of one or more real variables. It is true that the term "theory of functions of a real variable" calls to mind the idea of functions bristling with discontinuities or, if continuous, strewn with points where they have no derivative. The study of such functions has rendered a real service to mathematics during the last half century. There is, however, no more reason why an introduction to the study of functions of a real variable should plunge at once into such questions than there is in the theory of functions of a complex variable for turning at once to functions having non-analytic curves (or worse) as natural boundaries. M. Picard carefully avoided all questions of this nature in his first edition by making such assumptions as seem desirable in each case concerning continuity and existence of derivatives, even though by so doing he often failed to give his theorems the greatest possible generality. In this new edition he has still, in the main, pursued the same course; for the additions mentioned above which refer to discontinuous functions and the like are so brief as not seriously to divert the reader's attention from what may be called the cases of everyday life, while they still serve the useful purpose of calling his attention to the interesting possibilities which lie beyond.

In this volume the author makes no attempt at a systematic presentation of all parts of the subject and has thus been enabled to make it possible for the reader to open it at almost any chapter without being obliged to turn back too frequently to what has gone before. This is certainly a great advantage. I think, however, that in the treatment of uniform convergence, and of other questions which the word "uniform" suggests, the author has gone too far in this direction. The conception here involved is one of considerable difficulty in spite of its great simplicity, but it is one of such fundamental importance and constant application that it deserves to hold a central position in a treatise of this sort and to be used freely whenever occasion presents itself. The subject of the uniform convergence of series is admirably set forth in Chapter VIII, although it is to be regretted that Weierstrass's extremely simple and useful sufficient condition for uniform convergence is not explicitly mentioned.* It is, however, not pointed out that this is

* Namely that $\Sigma u_n(x)$ converges uniformly if a convergent series ΣM_n exists where $|u_n(x)| \leqq M_n$, the M's being positive constants. Weierstrass, Werke, vol. 2, p. 202.

merely a special case of the uniform convergence of a function involving several variables to a limit as some of these variables approach limits. Thus for instance we have in the question of the convergence of definite integrals precisely the same questions concerning uniform convergence as in the case of series.* All these questions, as well as the question of uniform continuity,† are not explicitly considered. Even in the case of series, however, M. Picard often seems to shrink from using the theorems he has established in Chapter VIII, and prefers to go over again in each special case what is practically the same proof he has already given in the general case. Thus on page 19 of Volume II, a mere glance at the series in line 6, written without a remainder, shows that it is uniformly convergent for all values of ψ and can therefore be integrated term by term. A precisely similar remark applies to page 57 of Volume II. Again on page 303–304 of Volume II. (the proof by successive approximations of the existence theorem for ordinary differential equations) the words uniform convergence do not appear, although this is the central idea involved.

There is no book on analysis better suited to the needs of graduate students in our universities than the treatise we are considering, avoiding as it does on the one hand the superficiality of almost all books on the subject in the English language, and on the other the heaviness and remoteness from other parts of mathematics characteristic of too many continental books on analysis which aim at giving a rigorous treatment.

MAXIME BÔCHER.

HARVARD UNIVERSITY,
November 2, 1901.

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* Cf. a paper by de la Vallée-Poussin: *Annales de la Société Scientifique de Bruxelles*, vol. 17 (1892-93), p. 323.

† The fundamental fact concerning uniform continuity for functions of two real variables is practically proved on p. 102, and states that if such a function is continuous within and on the boundary of a region it is uniformly continuous there. From this it follows that if r and ϑ are polar coördinates, and if $F(r, \vartheta)$ is continuous within and on the circumference of the circle $r=1$ then $F(r_1, \vartheta)$ $(r_1<1)$ approaches its limit $F(1, \vartheta)$ uniformly as r_1 approaches the value 1. This is the theorem proved in a special case by means of two very special inequalities on p. 276. It is also essentially identical with the fact referred to at the bottom of p. 42.

ERRATA.

The following errors in recent numbers of the BULLETIN have come to the attention of the editors:

(1) Mr. J. K. Whittemore advises us that the statement made in his review of Bianchi's Differential Geometry in the BULLETIN for July, 1901, page 438, lines 7–10, is incorrect and should be omitted from the review.

(2) In the present volume, on page 22, line 8, for $x + py = x_1$ read $x + p(y) = x_1$.

NOTES.

THE Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY will this year be extended to cover two days, Friday and Saturday, December 27–28. The Council will meet on Friday morning, and the annual election of officers and other members of the Council will be held on Saturday morning. Under a recent amendment of the By-Laws, the presidential address will be postponed to the annual meeting of 1902, the end of the presidential term.

THE date of the tenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY has been changed to Thursday and Friday, January 2–3, 1902. The meeting will be held at Evanston, as previously announced.

A NEW edition of the Annual Register of the Society will be issued in January next. Forms for furnishing necessary information have been sent to each member. Those who have not already responded to the circular are requested to do so as early as possible.

AT the recent annual meeting of the London mathematical society the following officers were elected for the ensuing year: President, Dr. E. W. HOBSON; vice-presidents, Professor W. BURNSIDE and Major MACMAHON; treasurer, Dr. J. LARMOR; honorary secretaries, Mr. R. TUCKER and Professor A. E. H. LOVE; also the following members of the council: Mr. J. E. Campbell; Lieut.-Col. A. J. C. Cunningham, Professor E. B. Elliott, Dr. J. W. L. Glaisher, Professor M. J. M. Hill, Mr. H. M. Macdonald, Professor L. J. Rogers, Mr. A. E. Western, Mr. E. T. Whittaker, and Mr. A. Young.

At a meeting of the Cambridge philosophical society on October 28, the following papers were read: Dr. E. W. Hobson: "Some remarks on the notion of number"; Mr. G. T. Walker: "On some problems in electric convection." At the meeting of November 25 the following papers were read: Major P. A. MacMahon: "On the seminvariants of systems of binary quantics, the order of each quantic being infinite"; Mr. J. H. Grace: "On the zeros of polynomials."

The German society of naturalists and physicians will hold its next annual meeting at Carlsbad.

The eleventh Congress of Russian naturalists and physicians will be held in St. Petersburg January 2–12, 1902. Of the thirteen sections of the congress three will be devoted to the mathematical sciences, namely, mathematics and mechanics, astronomy and geodesy, physics.

At the last annual meeting of the British association for the advancement of science a committee, with Professor A. R. Forsyth as president and Professor John Perry as secretary, was appointed to prepare a report on improvements that might be effected in the teaching of mathematics.

The National society of the natural and mathematical sciences of Cherbourg completes with the current year the fiftieth year of its existence.

The international association for promoting the study of quaternions and allied systems of mathematics has recently issued its annual bulletin. The president of the association for the year 1902 is Professor C. J. Joly, of the University of Dublin; the general secretary and treasurer is Dr. Alexander Macfarlane, of Lehigh University. There are also ten national secretaries. The association has at present about seventy members.

Beginning with its eleventh volume (1902), the *Jahresbericht* of the German Mathematical Society will enter upon a greatly enlarged sphere of activity under the continued editorship, now made permanent, of Professor Dr. A. Gutzmer, of Jena. In the future the *Jahresbericht* will appear in the form of a monthly periodical, containing as at present, reports of the annual meetings of the Society with abstracts of the papers, list of members and an administrative record. More elaborate papers and reports will be issued in special volumes. In addition, the enlarged plans include the pub-

lication of academic addresses, notices of proposed mathematical undertakings, the discussion of university systems of instruction, reports of meetings of national or international societies and associations, lists of courses of lectures, notices of the work of academies and societies, lists of publications, reviews and personal news. A full representation of foreign mathematical interests is greatly desired and to this end the editor asks the assistance and coöperation of mathematicians throughout the world. Communications should be addressed to Professor Dr. A. Gutzmer, 4 Schaefferstrasse, Jena, Germany.

Each volume of the *Jahresbericht* contains about 450 pages. The first (double) number of volume 11 will appear in the present month. The subscription price is eleven marks to members of the Society and fourteen marks to nonmembers. The Society has about 520 members. The annual dues are two marks, life membership fee thirty marks.

UNIVERSITY OF OXFORD.—The following lectures in mathematics are announced for the Michaelmas term, 1901, each course consisting of two lectures per week unless otherwise indicated:—By Professor E. B. ELLIOTT: Theory of numbers; substitutions and resolvents, one hour.—By Professor H. H. TURNER: Elementary mathematical astronomy.—By Professor W. ESSON: Analytic geometry of plane curves; synthetic geometry of plane curves, one hour.—By Professor A. E. H. LOVE: Introduction to mathematical physics; analytical dynamics, one hour. By Mr. C. E. HASELFOOT: Algebra.—By Mr. C. LEUDESDORF: Elementary projective geometry, three hours.—By Mr. C. H. SAMPSON.—Analytical geometry, one hour.—By Mr. J. W. RUSSELL: Differential calculus, one hour.—By Mr. P. J. KIRKBY: Introduction to higher algebra, one hour.—By Mr. A. L. PEDDER: Problems in pure mathematics.—By A. E. JOLLIFFE: Elementary solid geometry.—By Mr. J. E. CAMPBELL: Differential equations.—By Mr. C. H. THOMPSON: Integral calculus.—By Mr. E. H. HAYES: Analytical statistics.—By Mr. A. L. DIXON: Hydrostatics, one hour.—By Mr. H. T. GERRANS: Advanced rigid dynamics.

THE examiners for Part II of the Cambridge mathematical tripos for 1902 are Mr. HORACE LAMB, Mr. J. LARMOR, Mr. H. W. RICHMOND and Mr. E. T. WHITTAKER.

DURING his recent visit to the United States on the occasion of the bicentennial celebration of Yale University,

Professor J. HADAMARD, of the University of Paris, delivered lectures before various organizations at Columbia, Princeton, and Yale Universities. The lecture "On the mathematics of physics" before the mathematical seminary at Princeton will be published in the Princeton *Bulletin*.

THE Royal society of London has awarded its Copley medal to Professor JOSIAH WILLARD GIBBS, of Yale University, for his contributions to mathematical physics.

DR. E. NEUMANN, docent in mathematics at the University of Halle, has been appointed assistant professor of physics at the University of Breslau.

DR. G. KOWALEWSKI, docent in mathematics at the University of Leipsic, has been appointed assistant professor of mathematics at the University of Greifswald, and member of the examining commission. Dr. J. SOMMER, of Göttingen, has been appointed professor of mathematics in the agricultural school at Bonn-Poppelsdorf.

MR. R. P. PARANJPYE, bracketed senior wrangler of Cambridge University in 1899, has been elected to a fellowship in St. John's College.

PROFESSOR C. B. WILLIAMS, of Kalamazoo College, has received leave of absence and is studying in Leipsic.

Dr. L. W. REID has been promoted to an assistant professorship of mathematics at Haverford College.

MR. W. D. LAMBERT has been appointed to an instructorship in mathematics at Purdue University.

DR. T. M. PUTNAM has been appointed to an instructorship in mathematics in the University of California.

DR. JOHN PURSER, for forty years professor of mathematics in Queen's College, Belfast, has retired.

A MEMORIAL meeting in honor of the late Professor HENRY AUGUSTUS ROWLAND was held at Johns Hopkins University on October 16th. The principal address, by Dr. T. C. MENDENHALL, is published in *Science*, volume 14, pages 865–877.

THE death is announced of Professor CHARLES A. BACON, professor of astronomy and director of the Smith observatory at Beloit College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLKE (G.). Die Komplementärflächen der pseudosphärischen Rotationsflächen und ihr Zusammenhang mit allgemeineren pseudosphärischen Flächen. (Diss., Halle.) Leipzig, Teubner, 1901. 8vo. 78 pp., 1 plate. M. 2.00

CAPELLI (A.). Lezioni sulla teoria delle forme algebriche. Napoli, 1901. 8vo. 303 pp. Fr. 10.00

DONDER (T. DE). Etudes sur les invariants intégraux. (Extrait des *Rendiconti del Circolo matematico di Palermo.*) Paris, Gauthier-Villars, 1901. 8vo. 66 pp. Fr. 3.00

FRIZZO (G.). De numeris libri duo authore Ioanne Noviomago, esposti e illustrati da G. Frizzo. Verona e Padova, Drucker, 1901. 8vo. 174 pp. Fr. 3.00

HERMITE (CHARLES), 1822–1901. Notice sur ses travaux scientifiques, par C. Jordan. Esquisse biographique et bibliographique par P. Mansion. Paris, Gauthier-Villars, 1901. 8vo. 47 pp. Fr. 1.00

JORDAN (C.). See HERMITE (C.).

LAURENT (H.). See STURM (R.).

MANSION (P.). See HERMITE (C.).

MILAN (P.). Aus dem Grenzgebiet zwischen Mathematik und Philosophie. Kiel, 1901. 4to. 38 pp.

MORETTI (A.). La superficie del Kummer studiata analiticamente colla trasformazione del Reye. Milano, Bernardoni, 1901. 8vo. 34 pp.

MORRICA (E.). Deformazione infinitesima delle evolute delle superficie pseudosferiche nella trasformazione complementare e di Bäcklund. Pisa, Giordano, 1901. 8vo. 27 pp.

NOVIOMAGUS (I.). See FRIZZO (G.).

PROUHET (E.). See STURM (R.).

SACHS (J.). Lehrbuch der projectivischen (neueren) Geometrie. (Synthetische Geometrie; Geometrie der Lage.) Nebst Sammlung gelöster und ungelöster Aufgaben. (3 Theile.) Theil II: Harmonische Gebilde; Entstehung der Kegelschnitte; Sätze von Pascal und Brianchon. Stuttgart, 1901. 8vo. 135 pp. M. 6.00

SAINT-GERMAIN (A. DE). See STURM (R.).

STRAZZERI (V.). Le eliche cilindriche. Sassari, Chiarella, 1901. 8vo. 34 pp.

STURM (R.). Cours d'analyse de l'Ecole polytechnique. Revu et corrigé par E. Prouhet et augmenté de la théorie élémentaire des fonctions elliptiques par H. Laurent. 12e édition, mise au courant des nouveaux programmes de la licence par A. de Saint-Germain. 2 volumes. Paris, Gauthier-Villars, 1901. 8vo. Fr. 15.00

II. ELEMENTARY MATHEMATICS.

CASTLE (F.). Practical mathematics for beginners. London and New York, Macmillan, 1901. 12mo. 9 + 313 pp. Cloth. $0.60

CHEMIN (O.). See PETERSEN (J.).

FANTASIA (P.). See KLIMPERT (R.).

GÉRARD (L.). Trigonométrie, à l'usage des candidats au baccalauréats de l'enseignement secondaire classique (deuxième partie, 2e série), moderne (première partie, deuxième partie, 2e et 3e séries). Paris, Société d'éditions scientifiques, 1901. (Manuels des baccalauréats.) Fr. 1.25

GIOVANETTI (G.). Sopra una equazione trasformata particolare di una equazione cubica completa. Pavia, Fusi, 1901. 8vo. 3 pp.

HOÜEL (J.). Tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques, suivies des logarithmes d'addition et de soustraction ou logarithmes de Gauss, et de diverses tables usuelles. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1901. 8vo. (Autorisée par décision ministérielle.) Fr. 2.00

——. Recueil de formules et de tables numériques, formant le complément des Tables de logarithmes à cinq décimales du même auteur. 3e édition, nouveau tirage. Paris, Gauthier-Villars, 1901. 8vo. Fr. 4.50

KLIMPERT (R.). Storia della geometria ad uso dei dilettanti di matematica e degli alunni delle scuole secondarie. Traduzione dal tedesco autorizzata dall'autore con note ed aggiunte di P. Fantasia. Bari, Laterza, 1901. 8vo. 324 + 10 pp. Fr. 4.00

MARREL (E.). Théorie d'arithmétique dans l'enseignement secondaire. Lausanne, 1901. 8vo. 91 pp. Fr. 3.00

PETERSEN (J.). Méthodes et théories pour la résolution des problèmes de constructions géométriques, avec application à plus de 400 problèmes. Traduit par O. Chemin. 3e édition. Paris, Gauthier-Villars, 1901. 8vo. Fr. 4.00

PHILLIPS (E. A.). Algebra for beginners. London, Simpkin, 1901. 8vo. 1s. 6d.

REISCH (A.). In wiefern sind die Reformbewegungen beim einleitenden geometrischen Unterricht berechtigt? Chemnitz, 1901. 4to. 32 pp.

SMITH (H. W. C.). Primer of geometry. Comprising the subject-matter of Euclid I–IV, treated by the methods of pure geometry. London and New York, Macmillan, 1901. 12mo. 16 + 100 pp. Cloth. $0.50

SPERANDEI (L.). Aritmetica, sistema metrico decimale e geometria per il corso elementare superiore: classe quarta. Lanciano, Carabba, 1901. 16mo. 63 pp. Fr. 0.30

——. Classe quinta. Lanciano, Carabba, 1901. 16mo. 55 pp. Fr. 0.30

TARTINVILLE (A.). Cours d'arithmétique. 3e édition. Paris, Nony, 1902. 8vo. 519 pp.

III. APPLIED MATHEMATICS.

ABBOTT (T. K.). Teorica elementare delle maree e discussione della influenza che esercitano sulla durata del giorno. Versione italiana da E. de Ferrari. Pestoia, 1901. 8vo. 63 pp. Fr. 2.00

BEYEL (C.). Darstellende Geometrie. Mit einer Sammlung von 1800 Dispositionen zu Aufgaben aus der darstellenden Geometrie. Leipzig, Teubner, 1901. 8vo. 12 + 189 pp., 1 plate. Cloth. M. 3.60

BREGGEN (J. VAN DER). Leerboek der mechanica, met vele opgeloste vraagstukken en opgaven. Groningen, 1901. 8vo. 8 + 192 pp. M. 2.00

CAVALLI-LANFREDI (R.). Due casi nuovi di moto di fluidi; nota. Milano, Bernardoni, 1901. 8vo. 8 pp.

DARWIN (G. H.). Tides and kindred phenomena in the solar system. 2d edition. London, 1901. 8vo. 365 pp. Cloth. 7s. 6d.

FERRARI (E. DE). See ABBOTT (T. K.).

GIORLI (E.). Disegno, teoria e costruzione della nave. Milano, 1901. 12mo. 6 + 238 pp. Fr. 2.50

GUTHE (K. E.). Ueber die Funkenentladung bei schnellen Oscillationen. Leipzig, Barth, 1901. 8vo. (*Annalen der Physik* (4) 5, pp. 818–828.)

HERNANDEZ (P. G.). Teoria elemental del potencial eléctrico. Madrid, 1901. 8vo. 4 + 111 pp. Fr. 2.50

KORDGIEN (H.). Mathematisch-physikalische Aufgaben-Sammlung für die oberen Klassen höherer Lehranstalten, mit ausführlichen Erläuterungen und Lösungen. Teil 1: Mechanik. Berlin, Grote, 1901. 8vo. 7 + 147 pp. M. 2.20

KORN (A.). Abhandlungen zur Potentialtheorie. Heft 3: Ueber die zweite und dritte Randwertaufgabe und ihre Lösung. Berlin, Dümmler, 1901. 8vo. 56 pp. M. 1.00

——. Heft 4: Ueber die Differentialgleichung $\Delta U + \kappa\phi^2 U = f$ und die harmonischen Funktionen Poincarés. Berlin, Dümmler, 1902. 8vo. 55 pp. M. 1.00

MARCHIS (L.). Leçons sur les moteurs à gaz et à pétrole faites à la Faculté des sciences de Bordeaux. Paris, Gauthier-Villars, 1901. 16mo. 175 pp. Fr. 2.75

PAPPERITZ (E.). Ueber die wissenschaftliche Bedeutung der darstellenden Geometrie und ihre Entwicklung bis zur systematischen Begründung durch P. Monge. Freiberg, 1901. 8vo. M. 1.00

PASQUIER (E.). Cours de mécanique analytique. Vol. I: Vecteurs; Cinématique; statique et dynamique du point. Paris, Gauthier-Villars, 1901. 8vo. 358 pp. Fr. 12.00

ROSENBERG (F.). First stage mechanics of solids, for elementary examination of Science and art department. 4th edition. London, Clive, 1901. 8vo. 328 pp. 2s.

THOMPSON (S. P.). Courants polyphasés et alterno-moteurs. Traduction par E. Boistel. 2e édition. Paris, 1901. Fr. 25.00

WEINSTEIN (B.). Einleitung in die höhere mathematische Physik. Berlin, Dümmler, 1901. 8vo. 16 + 399 pp. Cloth. M. 7.00

WERNICKE (AD.). Lehrbuch der Mechanik in elementarer Darstellung, mit Anwendungen und Uebungen aus den Gebieten der Physik und Technik. Teil 1: Mechanik fester Körper, herausgegeben von Al. Wernicke. 4te Auflage. Abteilung 2: Statik und Kinetik des starren Körpers. Braunschweig, Vieweg, 1901. 8vo. Pp. 315–809. M. 6.00